§1 Functorial viewpoint

k: 体.

$f(X_1, \dots, X_n) \in k[X] \rightsquigarrow \mathrm{Sol}(f, k) := \{(x_1, \dots, x_n) \in k^n \mid f(x_1, \dots, x_n) = 0\}$

($X_1, \dots, X_n$ =: X)

· $\mathrm{Alg}(k[X]/(f(X)), k) := \{\varphi : k[X]/(f(X)) \to k : \text{algebraic maps}\}$

· $\mathrm{Spec}(k[X]/(f(X))) \longleftrightarrow \mathrm{Alg}(k[X]/(f(X)), -) : (k\text{-alg}) \longrightarrow (\mathcal{S}et)$

local ringed space　　　　functor

" geometric viewpoint"　　　　" functorial viewpoint"

· $A \in k\text{-alg}$.

$F := \mathrm{Sp}(A) = \mathrm{Alg}(A, -) : (k\text{-alg}) \longrightarrow (\mathcal{S}et)$

$B \longmapsto \mathrm{Alg}(A, B)$

$B \in k\text{-alg}$, $I \in \mathrm{ideal}(A)$ に対して

$$V(I)(B) := \{\varphi \in F(B) \mid \varphi(I) = 0\}$$

を考える. functor $V(I)(-) \subset \mathrm{Sp}(A)$ を closed subfunctor という.
また,

$$D(I)(B) := \{\psi \in F(B) \mid \psi(I)B = B\}$$

を考え, functor $D(I)(-) \subset \mathrm{Sp}(A)$ を open subfunctor という.

· 一般に, k-functor F の subfunctor G が

- closed $\overset{\text{def}}{\Longleftrightarrow}$ $^\forall f : \mathrm{Sp}(A) \Longrightarrow F$: natural trans., $f^{-1}(G)$ が $\mathrm{Sp}(A)$ の closed subfunctor.

$$\begin{array}{ccc} \mathrm{Sp}(A)(R) & \xrightarrow{f_R} & F(R) \\ \uparrow & \circlearrowleft & \uparrow \\ f_R^{-1}(G(R)) & \longrightarrow & G(R) \end{array}$$

closed sub

- open $\overset{\text{def}}{\Longleftrightarrow}$ $^\forall f : \mathrm{Sp}(A) \Longrightarrow F$: natural. trans., $f^{-1}(G)$ が $\mathrm{Sp}(A)$ の open sub.

・$\{F_i \ (i \in I)\}$: F の open subfunctors が F の **open covering**

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall\, \Bbbk \subset K$: field extension, $F(K) = \bigcup_{i \in I} F_i(K)$

・F が **local** $\overset{\text{def}}{\Longleftrightarrow}$ $\forall G$: $\Bbbk$-functor, $\{G_i\}$: open covering of G

$$\mathrm{Mor}_{\Bbbk\text{-func}}(G,F) \longrightarrow \prod_{i \in I} \mathrm{Mor}_{\Bbbk\text{-func}}(G_i,F) \rightrightarrows \prod_{i,j \in I} (G_i \cap G_j, F) \ \text{: exact}$$

$$f \longmapsto \prod_{i \in I} f|_{G_i} \quad \begin{matrix} \longmapsto \prod f_i|_{G_i \cap G_j} \\ \longmapsto \prod f_j|_{G_i \cap G_j} \end{matrix}$$

F が local かつ $\exists \{F_i\}_{i \in I}$: open covering of F (s.t.) $F_i = \mathrm{Sp}(A_i)$, ($A_i \in \Bbbk$-alg)
のとき, F を **functorial scheme** と呼ぶ.

$\Bbbk$-functor ‐‐‐‐adjointness‐‐‐→ local ringed space
↑ ↑
functorial scheme ∋ $\mathrm{Mor}(X,-) \longleftrightarrow X$ ∈ geometric scheme
functorial scheme ≈ (cat. equiv.) geometric scheme
↑ ↑
functorial affine scheme ≈ (cat. equiv.) geometric affine scheme

§2 Super symmetry

$\mathrm{char}(\Bbbk) \neq 2$ と仮定する.

$\mathcal{SV}$: the cat. of super vector spaces $\overset{\text{def}}{\Longleftrightarrow}$

・objects : $V = V_0 \oplus V_1$ ($\mathbb{Z}_2$-graded vect. sp.)

・morphisms : $f : V \longrightarrow W$ ($\mathbb{Z}_2$-graded map.)

・tensor prod. : $(V \otimes W)_l := \bigoplus_{\substack{i+j=l \\ (\mathrm{mod}\ 2)}} V_i \otimes W_j$

・unit : $\Bbbk \oplus 0$

・super symmetry : $C_{V,W} : V \otimes W \longrightarrow W \otimes V$

$$v \otimes w \longmapsto \begin{cases} -w \otimes v, & \text{if } v \in V_1,\ w \in W_1 \\ w \otimes v, & \text{otherwise} \end{cases}$$

⇝ $\mathcal{SV}$ は 対称テンソル圏 となる.
通常のベクトル空間 は $V \oplus 0$ として $\mathcal{SV}$ の object と 思う.

Theorem [Deligne]

$\mathrm{Char}(k) = 0$, $k = \bar{k}$ とする.

このとき, 任意のリジットな対称テンソル圏で, ある条件を満たすものは, ある algebraic super group 上の有限次元加群圏として実現される.

・ $\mathbb{H} = (H, m, u, \Delta, \varepsilon, S)$: Hopf algebra

$\overset{\text{def}}{\Longleftrightarrow}$ $H \otimes H \underset{m}{\overset{\Delta}{\rightleftarrows}} H \underset{u}{\overset{\varepsilon}{\rightleftarrows}} k$ が (Δ: 余積, ε: 余単位射, m: 積, u: 単位射, $H \circlearrowleft S$: 対合)

・ (H, m, u) : k-algebra

・ (H, Δ, ε) : k-coalgebra

・ Δ, ε : algebra map

・ $m \circ (1_H \otimes S) \circ \Delta = m \circ (S \otimes 1_H) \circ \Delta = u \circ \varepsilon$.

Hopf 代数 $\mathbb{H}$ が super commutative $\overset{\text{def}}{\Longleftrightarrow}$ $H \otimes H \xrightarrow{C_{H,H}} H \otimes H$, $m \searrow\ \circlearrowleft\ \swarrow m$, H

super co-comutative $\overset{\text{def}}{\Longleftrightarrow}$ $H \otimes H \xrightarrow{C_{H,H}} H \otimes H$, $\Delta \nwarrow\ \circlearrowleft\ \nearrow \Delta$, H

Hopf 代数 $\mathbb{H}$ の $m, u, \Delta, \varepsilon, S$ が $\mathbb{Z}_2$-graded mor のとき, super Hopf alg. という.

Example

V: vector space

$V = 0 \oplus V$ とみなす. $\rightsquigarrow$ $\Lambda(V)$: Vの外積代数 $= \bigoplus_{n=0}^{\infty} \Lambda^n(V)$: $\mathbb{N}$-graded algebra

$$= \underbrace{\left(\bigoplus_{n=0}^{\infty} \Lambda^{2n}(V) \right)}_{\text{even}} \oplus \underbrace{\left(\bigoplus_{n=0}^{\infty} \Lambda^{2n+1}(V) \right)}_{\text{odd}}$$

・ $v_1 \wedge v_2 = -v_2 \wedge v_1$

・ $\Delta(v) := v \otimes 1 + 1 \otimes v$

) super com. & super co-com.

§3 Formal super group.

Definition [Demazure]

$\mathbb{H}$: co-commutative super Hopf alg.

$$Sp^*(\mathbb{H}) : \underset{\text{super comm. super alg.}}{(SAlg)} \longrightarrow (Grp)$$

$$R \longmapsto Sp^*(\mathbb{H}) := \left\{ g \in (R \otimes H)_0 \;\middle|\; \begin{array}{l} \Delta_R(g) = g \otimes g \\ \varepsilon_R(g) = 1_R \end{array} \right\}$$

$$\left(\begin{array}{ccc} R \otimes H & \xrightarrow{\Delta_R} & (R \otimes H) \otimes_R (R \otimes H) \\ \cup & & \| \\ (r, h) & & R \otimes H \otimes H \\ & \searrow & \cup \\ & & r \otimes \Delta(h) \end{array} \right)$$

unit も同様

この $Sp^*(\mathbb{H})$ を formal supergroup という.

Example

$\mathfrak{g}$: super Lie alg.

$U(\mathfrak{g})$: the universal enveloping alg (super Hopf alg)

$$\left(\begin{array}{l} \Delta(x) = 1 \otimes x + x \otimes 1 \\ \varepsilon(x) = 0 \end{array} \right)$$

$\dim(\mathfrak{g}) < \infty$ とする.

$Sp^*(U(\mathfrak{g}))(R) \subset R \otimes U(\mathfrak{g})$ において

$$\left\{ \begin{array}{ll} \exp(\varepsilon \otimes x) = 1 \otimes 1 + \varepsilon \otimes x, & \varepsilon \in R_0,\ \varepsilon^2 = 0,\ x \in \mathfrak{g}_0 \\ \exp(a \otimes v) = 1 \otimes 1 + a \otimes v, & a \in R_1,\ v \in \mathfrak{g}_1 \end{array} \right.$$

全体で生成される部分群を $\Sigma(R)$ とおくと,

$$\Sigma(R) \supset \langle \exp(\varepsilon \otimes x) \rangle \longrightarrow G_0(R_0) \quad \cdots\cdots\cdots (1)$$

G_0 は代数群で $Lie(G_0) = \mathfrak{g}_0$ とする.

Theorem [Masuoka − Shibata]

$\Sigma(R)$ を (1)の射の延長として 群 $P(R)$が得られ, $R \longmapsto P(R)$ は algebraic super group.
すべての algebraic super group は この構成で得ることができる.

Theorem [T − Masuoka]

$\mathbb{J}$: super co-commutative Hopf super alg.
$\cup$
$\mathbb{K}$: sub Hopf super algebra

このとき,

$$\mathbb{J}/\mathbb{J}\mathbb{K}^+ \simeq J \otimes_K \wedge(Q) \text{ as super right } \mathbb{K}\text{-mod, coalg}.$$

対応 $\updownarrow$

$$Sp^*(\mathbb{J})/Sp^*(\mathbb{K})$$

ここで,

$\mathbb{K}^+ := \ker(\varepsilon_{\mathbb{K}})$, $J = \Delta^{-1}_{\mathbb{J}}(\mathbb{J}_0 \otimes \mathbb{J}_0)$: (super でない) Hopf alg

$K := \Delta^{-1}_{\mathbb{K}}(\mathbb{K}_0 \otimes \mathbb{K}_0)$, $Q = P(\mathbb{J})/P(\mathbb{K})$, $P(X) = \{x \in X \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$

cf. "Quotients in super symmetry formal super group case"