

§1 Functorial viewpoint

\mathbb{k} : 体

$f(x_1, \dots, x_n) \in \mathbb{k}[X] \rightsquigarrow \text{Sol}(f, \mathbb{k}) := \{ (x_1, \dots, x_n) \in \mathbb{k}^n \mid f(x_1, \dots, x_n) = 0 \}$

$\cdot \text{Alg}(\mathbb{k}[X]/(f(x)), \mathbb{k}) := \{ \varphi: \mathbb{k}[X]/(f(x)) \rightarrow \mathbb{k} : \text{algebraic maps} \}$

$\cdot \text{Spec}(\mathbb{k}[X]/(f(x))) \longleftrightarrow \text{Alg}(\mathbb{k}[X]/(f(x)), -) : (\mathbb{k}\text{-alg}) \rightarrow (\text{Set})$
local ringed space function

"geometric viewpoint"

"functorial viewpoint"

$\cdot A \in \mathbb{k}\text{-alg}$

$F := \text{Sp}(A) = \text{Alg}(A, -) : (\mathbb{k}\text{-alg}) \rightarrow (\text{Set})$
 $B \mapsto \text{Alg}(A, B)$

$B \in \mathbb{k}\text{-alg}$, $I \in \text{ideal}(A) = \mathcal{I} \subseteq \mathbb{k}[X]$

$V(I)(B) := \{ \varphi \in F(B) \mid \varphi(I) = 0 \}$

を考えると, functor $V(I)(-) \subset \text{Sp}(A)$ は closed subfunctor になる.

また,

$D(I)(B) := \{ \varphi \in F(B) \mid \varphi(I)B = B \}$

を考えると, functor $D(I)(-) \subset \text{Sp}(A)$ は open subfunctor になる.

\cdot 一般に, \mathbb{k} -functor F の subfunctor G が:

- closed $\stackrel{\text{def}}{\iff} \forall f: \text{Sp}(A) \rightrightarrows F : \text{natural trans.}, f^{-1}(G)$ が $\text{Sp}(A)$ の closed subfunctor.

$$\begin{array}{ccc} \text{Sp}(A)(R) & \xrightarrow{f_R} & F(R) \\ \uparrow & \cup & \uparrow \\ \text{---} f_R^{-1}(G(R)) \text{---} & \longrightarrow & G(R) \\ \text{closed sub} & & \end{array}$$

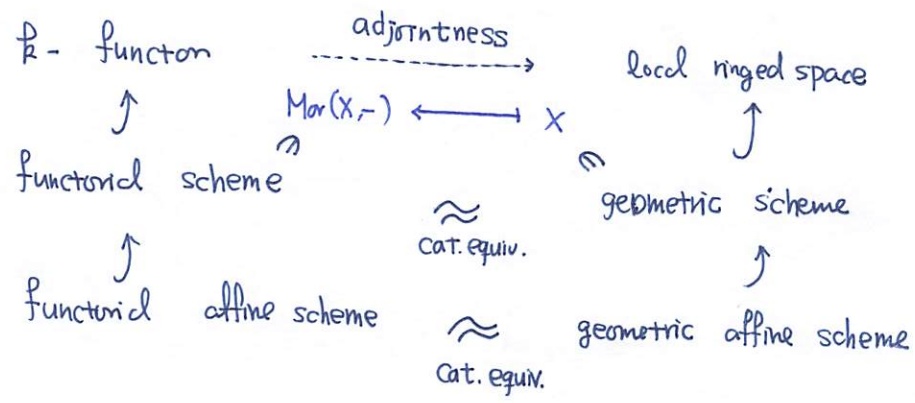
- open $\stackrel{\text{def}}{\iff} \forall f: \text{Sp}(A) \rightrightarrows F : \text{natural trans.}, f^{-1}(G)$ が $\text{Sp}(A)$ の open sub.

• $\{F_i (i \in I)\}$: F の open subfunctors が F の **open covering**
 $\stackrel{\text{def}}{\iff} \mathbb{R} \subset^V K$: field extension, $F(K) = \bigcup_{i \in I} F_i(K)$

• F が **local** $\stackrel{\text{def}}{\iff} \forall G: \mathbb{R}\text{-functor}, \{G_i\}$: open covering of G

$$\begin{aligned} \text{Mor}_{\mathbb{R}\text{-func}}(G, F) &\longrightarrow \prod_{i \in I} \text{Mor}_{\mathbb{R}\text{-func}}(G_i, F) \rightrightarrows \prod_{i, j \in I} (G_i \cap G_j, F) : \text{exact} \\ f &\longmapsto \prod_{i \in I} f|_{G_i} \longmapsto \prod f_i|_{G_i \cap G_j} \\ &\longmapsto \prod f_j|_{G_i \cap G_j} \end{aligned}$$

F が local かつ $\exists \{F_i\}_{i \in I}$: open covering of F (s.t.) $F_i = \text{Sp}(A_i)$, ($A_i \in \mathbb{R}\text{-alg}$)
 のとき, F を **functorial scheme** と呼ぶ。



§2 Super symmetry

$\text{char}(\mathbb{R}) \neq 2$ と仮定する。

- SV : the cat. of super vector spaces $\stackrel{\text{def}}{\iff}$
- objects: $V = V_0 \oplus V_1$ (\mathbb{Z}_2 -graded vect. sp.)
 - morphisms: $f: V \rightarrow W$ (\mathbb{Z}_2 -graded morph.)
 - tensor prod.: $(V \otimes W)_\ell := \bigoplus_{\substack{i+j=\ell \\ (\text{mod } 2)}} V_i \otimes W_j$
 - unit: $\mathbb{R} \oplus 0$
 - Super symmetry: $C_{V,W}: V \otimes W \rightarrow W \otimes V$
 $v \otimes w \mapsto \begin{cases} -w \otimes v, & \text{if } v \in V_1, w \in W_1 \\ w \otimes v, & \text{otherwise} \end{cases}$

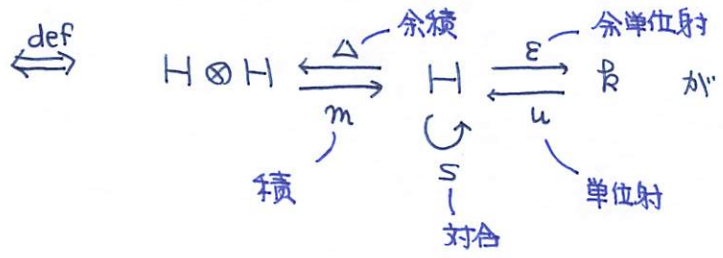
$\rightsquigarrow SV$ は 対称テンソル圏 と呼ぶ。
 通常のベクトル空間は $V \oplus 0$ として SV の object と見ろ。

Theorem [Deligne]

$\text{Char}(\mathbb{k}) = 0, \mathbb{k} = \overline{\mathbb{k}}$ とする.

このとき, 任意の有限次元対称テンソル圏で, ある条件を満たすものは, ある algebraic super group 上の有限次元加群圏として実現される.

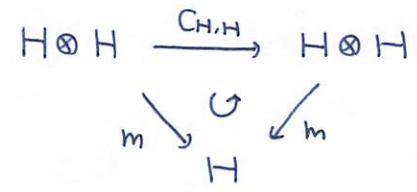
• $H = (H, m, u, \Delta, \varepsilon, S) : \text{Hopf algebra}$



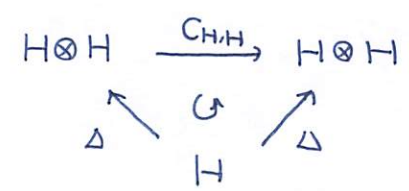
- $(H, m, u) : \mathbb{k}\text{-algebra}$
- $(H, \Delta, \varepsilon) : \mathbb{k}\text{-coalgebra}$
- $\Delta, \varepsilon : \text{algebra map}$

• $m \circ (1_H \otimes S) \circ \Delta = m \circ (S \otimes 1_H) \circ \Delta = u \circ \varepsilon$

Hopf代数 H が **super commutative** $\stackrel{\text{def}}{\iff}$



Super co-commutative $\stackrel{\text{def}}{\iff}$



Hopf代数 H の $m, u, \Delta, \varepsilon, S$ が \mathbb{Z}_2 -graded mor のとき, **super Hopf alg.** といい.

Example

V : vector space

$V = 0 \oplus V$ とみ直す. $\rightsquigarrow \Lambda(V) : V$ の外積代数 $= \bigoplus_{n=0}^{\infty} \Lambda^n(V) : \mathbb{N}$ -graded algebra

$$= \underbrace{\left(\bigoplus_{n=0}^{\infty} \Lambda^{2n}(V) \right)}_{\text{even}} \oplus \underbrace{\left(\bigoplus_{n=0}^{\infty} \Lambda^{2n+1}(V) \right)}_{\text{odd}}$$

- $v_1 \wedge v_2 = -v_2 \wedge v_1$
 - $\Delta(v) := v \otimes 1 + 1 \otimes v$
- supercom. & super co-com.

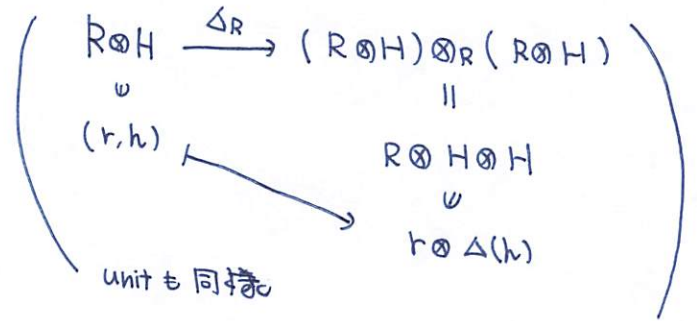
§3 Formal super group.

Definition [Demazure]

H : co-commutative super Hopf alg.

$$Sp^*(H) : \underbrace{(SAlg)}_{\substack{\text{super comm.} \\ \text{super alg.}}} \longrightarrow (Grp)$$

$$R \longmapsto Sp^*(H) := \left\{ g \in (R \otimes H)_0 \mid \begin{array}{l} \Delta_R(g) = g \otimes g \\ \epsilon_R(g) = 1_R \end{array} \right\}$$



$\therefore Sp^*(H)$ は formal supergroup となる。

Example

\mathfrak{g} : super Lie alg.

$U(\mathfrak{g})$: the universal enveloping alg (super Hopf alg)

$$\left(\begin{array}{l} \Delta(x) = 1 \otimes x + x \otimes 1 \\ \epsilon(x) = 0 \end{array} \right)$$

$\dim(\mathfrak{g}) < \infty$ とする。

$$Sp^*(U(\mathfrak{g}))(R) \subset R \otimes U(\mathfrak{g}) \text{ において}$$

$$\left\{ \begin{array}{l} \exp(\epsilon \otimes x) = 1 \otimes 1 + \epsilon \otimes x, \quad \epsilon \in R_0, \epsilon^2 = 0, x \in \mathfrak{g}_0 \\ \exp(a \otimes v) = 1 \otimes 1 + a \otimes v, \quad a \in R_1, v \in \mathfrak{g}_1 \end{array} \right.$$

全体を生成する部分群を $\Sigma(R)$ とおくと,

$$\Sigma(R) \supset \langle \exp(\epsilon \otimes x) \rangle \longrightarrow G_0(R_0) \text{ ----- (1)}$$

G_0 は 代数群で $Lie(G_0) = \mathfrak{g}_0$ とする。

Theorem [Masuoka - Shibata]

$\Sigma(R)$ を (1) の射の延長として 群 $P(R)$ が得られ, $R \mapsto P(R)$ は algebraic super group.
 全ての algebraic super group は この構成で得ることができる.

Theorem [T - Masuoka]

\mathcal{J} : super co-commutative Hopf super alg.

\cup

\mathcal{K} : sub Hopf super algebra

このとき,

$$\begin{array}{ccc} \mathcal{J} / \mathcal{J}\mathcal{K}^+ & \cong & \mathcal{J} \otimes_{\mathcal{K}} \wedge(\mathcal{Q}) \text{ as super right } \mathcal{K}\text{-mod, coalg.} \\ \downarrow \text{対応} & & \\ \text{Sp}^*(\mathcal{J}) / \text{Sp}^*(\mathcal{K}) & & \end{array}$$

ここで,

$$\mathcal{K}^+ := \ker(\varepsilon_{\mathcal{K}}), \quad \mathcal{J} = \Delta_{\mathcal{J}}^{-1}(\mathcal{J}_0 \otimes \mathcal{J}_0) : (\text{super } \check{\tau}\text{-T}^u) \text{ Hopf alg}$$

$$\mathcal{K} := \Delta_{\mathcal{K}}^{-1}(\mathcal{K}_0 \otimes \mathcal{K}_0), \quad \mathcal{Q} = P(\mathcal{J}) / P(\mathcal{K}), \quad P(X) = \{x \in X \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

cf. "Quotients in super symmetry formal super group case"