パーフェクトイド理論のネーター化について

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第 7 回情報数理セミナー

2024 年 8 月 20 日（火）

1. Introduction

2. Perfectoid towers and their tilts
(j.w.w. Nakazato and Shimomoto : based on arXiv:2203.16400v3)

3. Perfectoid towers and Homological Conjecture
(j.w.w Ishizuka, Nakazato, and Shimomoto : in progress)

Definition (characteristic of local rings)

$(R, \mathfrak{m})$: a local ring

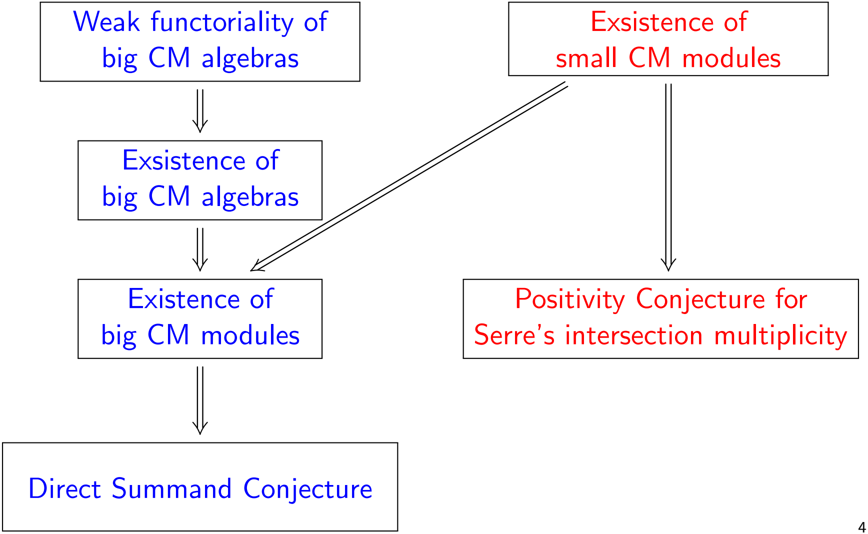
	$\operatorname{char} R$	$\operatorname{char} R/\mathfrak{m}$
equal char 0	0	0
equal char p	p	p
mixed char $(0, p)$	0	p
mixed char (p^n, p)	p^n	p

Example

1. $R = \mathbb{Q}[\![x_1, \ldots, x_d]\!]$ is a ring of characteristic 0.
2. $R = \mathbb{F}_p[\![x_1, \ldots, x_d]\!]$ is a ring of characterstic p.
3. $R = \mathbb{Z}_p[\![x_1, \ldots, x_d]\!]$ is a ring of characteristic $(0, p)$.
4. $R = (\mathbb{Z}/p^n\mathbb{Z})[\![x_1, \ldots, x_d]\!]$ is a ring of characteristic (p^n, p).

Homological Conjecture (1960s∼)

A series of conjectures for modules over a Noeth local ring



What are important tools in studies on commutative ring theory?

- Equal characteristic 0
  $\longrightarrow$ Mod p reduction, Resolution of singularities, etc.

- Equal characteristic p
  $\longrightarrow$ Frobenius homomorphism, Tight closure, etc.

- Mixed characteristic
  $\longrightarrow$???

What are important tools in studies on commutative ring theory?

- Equal characteristic 0
  $\longrightarrow$ Mod p reduction, Resolution of singularities, etc.

- Equal characteristic p
  $\longrightarrow$ Frobenius homomorphism, Tight closure, etc.

- Mixed characteristic
  $\longrightarrow$ **Perfectoid theory**.

Fix a prime number p.

Definition (Perfectoid rings)

R : a ring with $\varpi \in R$ s.t. $\varpi^p = pu$ for $u \in R^\times$

Suppose : R is ϖ-torsion free.

Then R is a **perfectoid ring** if

1. R is p-adically complete and separated (i.e. $\varprojlim_{n>0} R/p^n R \cong R$)
2. $F_{R/pR} : R/pR \to R/pR$ $(x \mapsto x^p)$ is surjective.
3. $\mathrm{Ker}\, F_{R/pR} = (\overline{\varpi})$.

Definition (Tilting)

For a (perfectoid) ring R, we define the **tilt** of R as

$$R^\flat := \varprojlim \{ \cdots \xrightarrow{F_{R/pR}} R/pR \xrightarrow{F_{R/pR}} R/pR \}.$$

Example

$\mathbb{Z}_p$: the ring of p-adic integers

$\widehat{\mathbb{Z}_p[p^{1/p^\infty}]} := \widehat{\bigcup_{n\geq 0} \mathbb{Z}_p[p^{1/p^n}]}$ (p-adic completion) is a perfectoid ring.

Moreover,

$$(\widehat{\mathbb{Z}_p[p^{1/p^\infty}]})^\flat \cong \widehat{\mathbb{F}_p[[t]][t^{1/p^\infty}]} = \widehat{\bigcup_{n\geq 0} \mathbb{F}_p[[t]][t^{1/p^n}]}$$

Example

$R := W(k)[[x_2, \ldots, x_d]]$

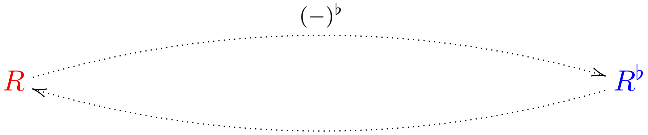
where $W(k)$: CDVR in mixed char with perfect residue field k.

$\widehat{R_\infty} := \widehat{\bigcup_{n\geq 0} R[p^{1/p^n}, x_2^{1/p^n}, \ldots, x_d^{1/p^n}]}$ is a perfectoid ring.

Moreover,

$$\widehat{R_\infty}^\flat \cong \widehat{\bigcup_{n\geq 0} k[[t, x_2, \ldots, x_d]][t^{1/p^n}, x_2^{1/p^n}, \ldots, x_d^{1/p^n}]}$$

Tilting correspondence



mixed char world

positive char world

$$\begin{Bmatrix}\text{Perfectoid rings} \\ \text{(mixed char)}\end{Bmatrix} \xrightarrow{(-)^\flat} \begin{Bmatrix}\text{Perfect(oid) rings} \\ \text{(positive char)}\end{Bmatrix} \longrightarrow \begin{Bmatrix}\text{Perfectoid rings} \\ \text{(mixed char)}\end{Bmatrix}.$$

Remark

Perfectoid rings and their tilts are **not** Noetherian.

Applications of perfectoid theory:

In arithmetic geometry,

- Weight-Monodoromy Conjecture (P. Scholze'12)
- Integral p-adic Hodge theory (Bhatt–Morrow–Scholze'19)
- Purity for flat (étale) cohomology (Cesnavicius–Scholze' 23)

In commutative ring theory,

- Homological Conjecture
  (Y. Andre'16,'20, B. Bhatt'16, Ma–Heitmann'17)
- Singularity theory in mixed characteristic
  (B. Bhatt, L. Ma, K. Schwede, etc. '17$\sim$)

Theorem (Cesnavicius–Scholze'23)

$(R, \mathfrak{m})$: a regular local ring,

G : a comm, finite, étale R-group whose order is invertible in R.
Then

$$H^i_{\mathfrak{m}}(R, G)_{\text{ét}} \cong 0 \qquad \text{for } i < 2\dim R.$$

The idea of the proof by Cesnavicius–Scholze

For simplicity, we may assume

$$R \cong W(k)[\![x_2, \ldots, x_d]\!]$$

where $W(k)$: CDVR and k: perfect.

$$\begin{array}{ccccccc} R & \longrightarrow & R[p^{1/p}, x_2^{1/p}, \ldots, x_d^{1/p}] & \longrightarrow & \cdots & \longrightarrow & \widehat{R_\infty} \\ & & & & & & \Big\downarrow {\scriptstyle (-)^\flat} \\ S & \longrightarrow & S[t^{1/p}, x_2^{1/p}, \ldots, x_d^{1/p}] & \longrightarrow & \cdots & \longrightarrow & \widehat{R_\infty}^\flat \end{array}$$

where $S := k[\![t, x_2, \ldots, x_d]\!]$.
It suffices to show : $H^i_{(x_1, \ldots, x_d)}(S, \mathbb{Z}/\ell\mathbb{Z})_{\text{ét}} \cong 0$,
and it's already known.

$$
\begin{array}{ccccccc}
R & \longrightarrow & R[p_1^{1/p}, \ldots, x_d^{1/p}] & \longrightarrow & \cdots & \longrightarrow & \widehat{R_\infty} \\
 & & & & & & \Big\downarrow{\scriptstyle (-)^\flat} \\
S & \longrightarrow & S[t^{1/p}, \ldots, x_d^{1/p}] & \longrightarrow & \cdots & \longrightarrow & \widehat{R_\infty}^\flat
\end{array}
$$

$R := W(k)[[x_2, \ldots, x_d]]$ (mixed char),

$S := k[[t, x_2, \ldots, x_d]]$ (positive char).

Our Motivation

$R_0 \to R_1 \to \cdots$: a direct system of Noeth local rings of mixed char,
$\widehat{R_\infty} := \widehat{\varinjlim_{n \geq 0} R_n}$: a perfectoid ring.
We want to construct a direct system $S_0 \to S_1 \to \cdots$ of char $p > 0$ s.t.

- $\widehat{S_\infty} := \widehat{\varinjlim_{n \geq 0} S_n} \cong \widehat{R_\infty}^\flat$, and
- S_n has the same properties as R_n.

Definition (Perfectoid towers) 1/2

R : a ring containing p, I_0 : an ideal of R.

Assume that R is I_0-torsion free.
$R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots$ is a **perfectoid tower arising from** (R, I_0).

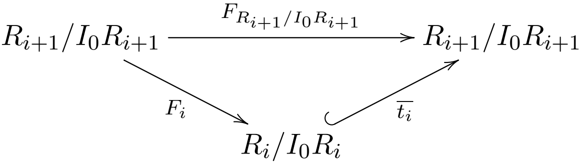
$\overset{\text{def}}{\Leftrightarrow}$ it's a direct system satisfying the following axioms (a)∼(f).

(a) $R_0 = R$ and $p \in I_0$,

(b) $\overline{t_i} : R_i/I_0R_i \to R_{i+1}/I_0R_{i+1}$ is injective,

(c) $\operatorname{Im} F_{R_{i+1}/I_0R_{i+1}} \subseteq \operatorname{Im} \overline{t_i}$.

By the axioms (b) and (c), we obtain the following factorization:



F_i is called the i-**th Frobenius projection**.

Definition (Perfectoid towers) 2/2

(d) F_i is surjective for any $i \geq 0$,

(e) R_i is I_0-adically complete and separated for any $i \geq 0$,

(f) I_0 is generated by a non-zero divisor, and
R_1 contains a principal ideal I_1 that satisfies the following.
  (f-1) $I_1^p = I_0 R_1$.
  (f-2) For every $i \geq 0$, $\mathrm{Ker}(F_i) = I_1(R_{i+1}/I_0 R_{i+1})$.

Definition (Small tilts)

$R_0 \to R_1 \to R_2 \to \cdots$: a perfectoid tower.

Then the j-**th small tilt** is defined as

$$R_j^{s.\flat} := \varprojlim \{ \cdots \xrightarrow{F_{j+1}} R_{j+1}/I_0 R_{j+1} \xrightarrow{F_j} R_j/I_0 R_j \}.$$

$R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots$: a perfectoid tower arising from (R_0, I_0).

Definition

We define

- $t_i^{s.\flat} : R_i^{s.\flat} \to R_{i+1}^{s.\flat} \ : \ (x_n)_{n\geq 0} \mapsto \left(\overline{t_{i+n}}(x_n)\right)_{n\geq 0}$.
- $I_0^{s.\flat} := \mathrm{Ker}(R_0^{s.\flat} \xrightarrow{\pi_0} R_0/I_0R_0)$.

Proposition

$R_0^{s.\flat} \xrightarrow{t_0^{s.\flat}} R_1^{s.\flat} \xrightarrow{t_1^{s.\flat}} \cdots$ is a perfectoid tower arising from $(R_0^{s.\flat}, I_0^{s.\flat})$

$(-)^{s.\flat}$

$R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots$ $\qquad$ $R_0^{s.\flat} \xrightarrow{t_0^{s.\flat}} R_1^{s.\flat} \xrightarrow{t_1^{s.\flat}} \cdots$

mixed char world $\qquad$ positive char world

Example (the ring of p-adic integers)

$R_0 := \mathbb{Z}_p,\ R_1 := \mathbb{Z}_p[p^{1/p}],\ R_2 := \mathbb{Z}_p[p^{1/p^2}],\ \cdots.$

Then

$$R_0 \xrightarrow{\subseteq} R_1 \xrightarrow{\subseteq} R_2 \xrightarrow{\subseteq} \cdots$$

is a perfectoid tower arising from $\big(\mathbb{Z}_p, (p)\big)$.

Moreover, $R_0^{s.\flat} \cong \mathbb{F}_p[\![t]\!]$ and $R_n^{s.\flat} \cong \mathbb{F}_p[\![t]\!][t^{1/p^n}]$.

$$\begin{array}{ccccccc}
\mathbb{Z}_p & \longrightarrow & \mathbb{Z}_p[p^{1/p}] & \longrightarrow & \cdots & \longrightarrow & Z_p\widehat{[p^{1/p^\infty}]} \\
\Big\downarrow (-)^{s.\flat} & & \Big\downarrow (-)^{s.\flat} & & & & \Big\downarrow (-)^{\flat} \\
\mathbb{F}_p[\![t]\!] & \longrightarrow & \mathbb{F}_p[\![t]\!][t^{1/p}] & \longrightarrow & \cdots & \longrightarrow & \mathbb{F}_p\widehat{[\![t]\!][t^{1/p^\infty}]}
\end{array}$$

Example (formal power series rings over CDVR)

$R_0 := W(k)[\![x_2, \ldots, x_d]\!]$ (k : perfect), $R_1 := R_0[p^{1/p}, x_2^{1/p}, \ldots x_d^{1/p}]$, ...
Then

$$R_0 \longrightarrow R_1 \longrightarrow R_2 \longrightarrow \cdots$$

is a perfectoid tower arising from $(R_0, (p))$.

Moreover, $R_0^{s.\flat} \cong k[\![t, x_2, \ldots, x_d]\!]$.

Example (Toric case : Gabber–Ramero, I.–Nakazato–Shimomoto)

$R := \mathbb{Z}_p[\![x, y, z]\!]/(px - yz)$,
$R_1 := \mathbb{Z}_p[\![p^{1/p}, x^{1/p}, y^{1/p}, z^{1/p}]\!]/(p^{1/p}x^{1/p} - y^{1/p}z^{1/p})$,
$R_2 := \mathbb{Z}_p[\![p^{1/p^2}, x^{1/p^2}, y^{1/p^2}, z^{1/p^2}]\!]/(p^{1/p^2}x^{1/p^2} - y^{1/p^2}z^{1/p^2})$.
Then

$$R_0 \longrightarrow R_1 \longrightarrow R_2 \longrightarrow \cdots$$

is a perfectoid tower arising from $(R_0, (p))$.

Moreover, $R_0^{s.\flat} \cong k[\![t, x, y, z]\!]/(tx - yz)$.

For a perfectoid tower $R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots$,
the projection $\pi_0 : R_0^{s.\flat} \twoheadrightarrow R_0/I_0$ induces

$$R_0^{s.\flat}/I_0^{s.\flat} \xrightarrow{\cong} R_0/I_0.$$

In particular,

$$\begin{array}{ccccc} R_0^{s.\flat} & & & & R_0 \\ & \searrow & & \swarrow & \\ & R_0^{s.\flat}/I_0^{s.\flat} & \xrightarrow{\cong} & R_0/I_0 & \end{array}$$

Proposition

$R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots$: a perfectoid tower arising from some $(R_0, (p))$.

1. R_i : local (resp. Noetherian) $\Leftrightarrow$ so is $R_i^{s.\flat}$.
2. If $p^{s.\flat}$ is a generator of $I_0^{s.\flat}$, then $p^{s.\flat}$ is NZD.
3. $\dim R_i = \dim R_i^{s.\flat}$.
4. R_i is CM (resp. Gorenstein) if and only if so is $R_i^{s.\flat}$.

Theorem

$R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots$: a perfectoid tower arising from some (R_0, I_0),

$\widehat{R_\infty}$: the I_0-adic completion of $R_\infty := \varinjlim R_i$,

$\widehat{R_\infty^{s.\flat}}$: the $I_0^{s.\flat}$-adic completion of $R_\infty^{s.\flat} := \varinjlim R_i^{s.\flat}$.

Then

1. $\widehat{R_\infty}$ is a perfectoid ring,
2. The tilt $(\widehat{R_\infty})^\flat$ is isomorphic to $\widehat{R_\infty^{s.\flat}}$.
   In particular,

$$\begin{array}{ccccccc} R_0 & \longrightarrow & R_1 & \longrightarrow & \cdots & \longrightarrow & \widehat{R_\infty} \\ \downarrow{\scriptstyle (-)^{s.\flat}} & & \downarrow{\scriptstyle (-)^{s.\flat}} & & & & \downarrow{\scriptstyle (-)^{\flat}} \\ R_0^{s.\flat} & \longrightarrow & R_1^{s.\flat} & \longrightarrow & \cdots & \longrightarrow & \widehat{R_\infty^{s.\flat}} \end{array}$$

Theorem (I.-Nakazato-Shimomoto)

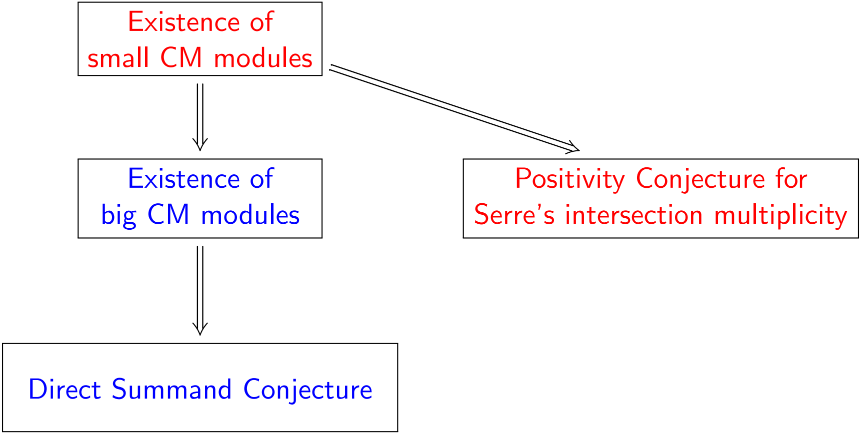
$R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots$: a perfectoid tower of Noeth normal domains,
ℓ : a prime different from p,
$U \subseteq \mathrm{Spec}(R)$: a Zariski-open subset s.t $\mathrm{Spec}(R) \setminus V(I_0) \subseteq U$
$U^{s.\flat} \subseteq \mathrm{Spec}(R^{s.\flat})$: the corresponding open subset
Assume : $t_j : R_j \to R_{j+1}$ is module-finite
whose generic extension is of p-power degree for every $j \geq 0$.
Then, for any fixed $i, n \geq 0$ such that $|H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})| < \infty$,

$$|H^i(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})| \leq |H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})|.$$

Question

1. Does the above inequality become the equality?
2. For a perfectoid tower $R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots$,
   is R_i a domain (resp. normal) $\Leftrightarrow$ so is $R_i^{s.\flat}$?

Homological Conjecture



Definition (Bhatt–Hochster–Ma)

R : a d-dim Noeth local ring,

$\{M_n\}_{n\geq 0}$: a sequence of f.g. R-modules with $\dim M_n = d$

Then $\{M_n\}_{n\geq 0}$ is a **lim Cohen–Macaulay sequence** if

$$\lim_{n\to\infty} \frac{h_i(\underline{x}; M_n)}{\nu(M_n)} = 0$$

for any system of parameters of R and any $i \geq 1$ where

$h_i(\underline{x}; M_n)$: **the length of the Koszul homology of M_n w.r.t $\underline{x}$**,

$\nu(M_n)$: **the least number of generators of M_n**.

Remark

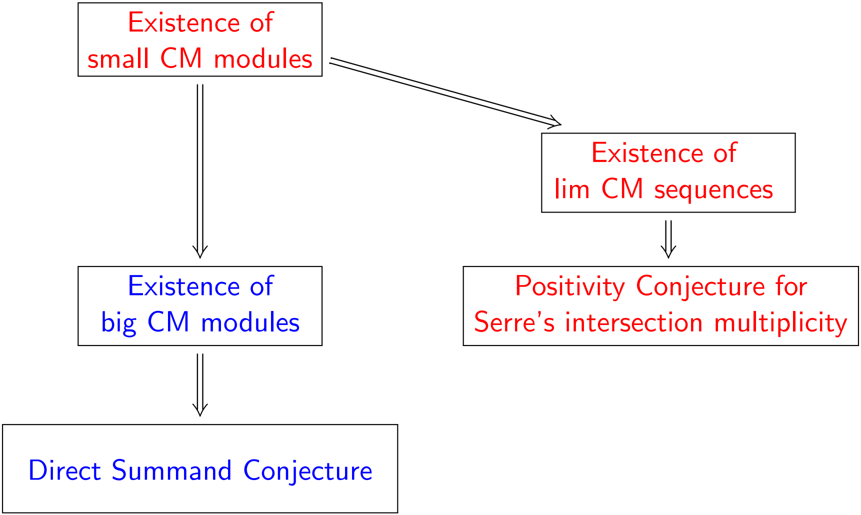
R : a Noetherian local ring, M : a small CM module

Then $\{M_n\}_{n\geq 0} = \{M\}_{n\geq 0}$ is a lim CM sequence of R.

Existence of small CM modules	$\Longrightarrow$	Existence of lim CM sequences

Theorem (Bhatt–Hochster–Ma)

If all complete local domains in mixed characteristic with perfect residue class fields have lim CM sequences, then the positivity conjecture is true.



Main Theorem (I.–Ishizuka–Nakazato–Shimomoto)

R : a complete Noetherian local domain with perfect residue field.

Suppose that

- there exists a perfectoid tower $R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots$ arising from (R_0, I_0) s.t. t_i is finite, and
- $R_0^{s.\flat}$ is a domain.

Then$\{R_n\}_{n\geq 0}$ is a lim Cohen–Macaulay sequence.

Lemma 1 (I.–Ishizuka–Nakazato–Shimomoto)

$(R, \mathfrak{m})$: d-dim Noetherian local ring,

$\{M_n\}_{n\geq 0}$: a seq of f.g. R-mod satisfying $d = \dim M_n$.

$x \in \mathfrak{m}$: a parameter element of R which is regular on M_n.

Then $\{M_n\}_{n\geq 0}$ is a lim Cohen–Macaulay sequence.

$\Leftrightarrow \{M_n/xM_n\}_{n\geq 0}$ is a lim Cohen–Macaulay sequence.

Lemma 2 (Bhatt–Hochster–Ma)

R : a complete Noeth local domain in char $p > 0$ with perfect residue field.
$R^{1/p^i} := \{a^{1/p^i} \mid a \in R\}$.
Then $\{R^{1/p^n}\}_{n \geq 0}$ is a lim CM sequence of R.

Proposition (I.–Nakazato–Shimomoto)

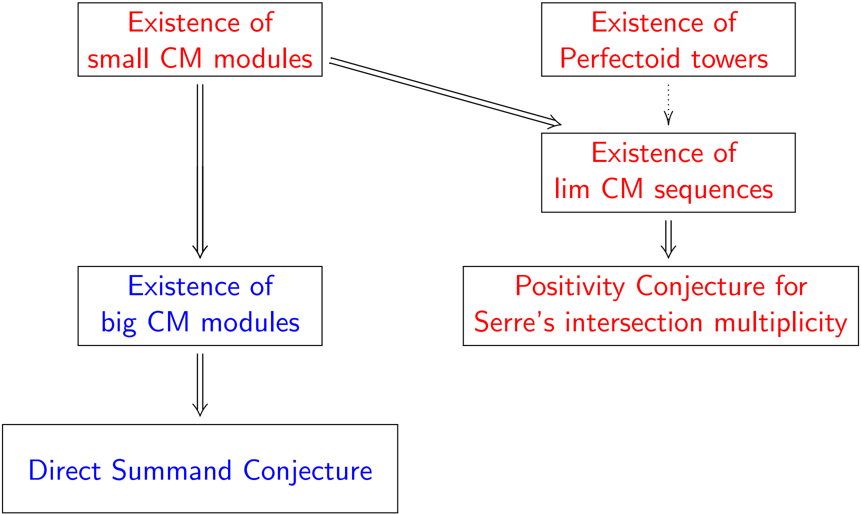
$R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots$: a perfectoid tower arising from some (R_0, I_0).

Then

1. $R_i^{s.\flat}$ is reduced.
2. $R_0^{s.\flat} \xrightarrow{t_0^{s.\flat}} R_1^{s.\flat} \xrightarrow{t_1^{s.\flat}} \cdots \quad \cong \quad R_0^{s.\flat} \hookrightarrow (R_0^{s.\flat})^{1/p} \hookrightarrow \cdots$.

   i.e.

$$\begin{array}{ccccccc} R_0^{s.\flat} & \xrightarrow{t_0^{s.\flat}} & R_1^{s.\flat} & \xrightarrow{t_1^{s.\flat}} & R_2^{s.\flat} & \xrightarrow{t_2^{s.\flat}} & \cdots \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ R_0^{s.\flat} & \longrightarrow & (R_0^{s.\flat})^{1/p} & \longrightarrow & (R_0^{s.\flat})^{1/p^2} & \longrightarrow & \cdots \end{array}$$



Question

Does any complete Noetherian local domain in mixed characteristic with perfect residue field have a perfectoid tower?