Super Lie groups and Harish-Chandra pairs.

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Notations

$\Bbbk$: field with $\mathrm{char}(\Bbbk) \neq 2$
Vec : the cat. of vec. sp.

	Vec	SVec : the cat. of super vector sp.
object :	vect. sp	$V = V_0 \oplus V_1$: $\mathbb{Z}_2$-grading
mor :	$\Bbbk$-linear map	graded $\Bbbk$-linear map
tensor :	$V \otimes_{\Bbbk} W$	$(V \otimes_{\Bbbk} W)_i = \bigoplus_{s+t=i} (V_s \otimes_{\Bbbk} W_t)$ $(i \in \mathbb{Z}_2)$
unit :	$\Bbbk$	$\Bbbk \oplus 0$
symmetry :	$V \otimes W \longrightarrow W \otimes V$, $v \otimes w \longmapsto w \otimes v$	Super symmetry : $V \otimes W \longrightarrow W \otimes V$, $v \otimes w \longmapsto (-1)^{\lvert w \rvert \lvert v \rvert} w \otimes v$

ただし, v, w は homogeneous で
$$\lvert v \rvert = \begin{cases} 0 & \text{if } v \in V_0 \\ 1 & \text{if } v \in V_1 \end{cases}, \quad \lvert w \rvert = \begin{cases} 0 & \text{if } w \in W_0 \\ 1 & \text{if } w \in W_1 \end{cases}.$$

Example

① $A = \bigoplus_{i=0}^{\infty} A_i$: $\mathbb{N} = \{0, 1, 2, \dots\}$-graded
$$\rightsquigarrow \Big(\bigoplus_{n:\mathrm{even}} A_n\Big) \oplus \Big(\bigoplus_{n:\mathrm{odd}} A_n\Big) \ : \text{super algebra } (= \mathbb{Z}_2\text{-graded alg})$$

具体的に $V = 0 \oplus V_1$ のとき, $T(V)$ は tensor alg.

② $\mathbb{Z}_2$-graded alg $A = A_0 \oplus A_1$ に対して, homogen. ideal のことを super ideal と呼ぶ. //

Definition

H が $\Bbbk$ 上の Hopf alg であるとは,
$$\Bbbk \underset{\varepsilon}{\overset{u}{\rightleftarrows}} H \underset{\Delta}{\overset{m}{\leftrightarrows}} H \otimes H, \qquad H \overset{S}{\circlearrowleft}$$
が以下の条件をみたす:

[H1] (H, m, u) alg.
[H2] Δ, ε : alg. map.
[H3] (H, Δ, ε) coalg.
[H4]
$$\begin{tikzcd}
H \otimes H \arrow[d, "S \otimes \mathrm{id}"'] & H \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\varepsilon"] & H \otimes H \arrow[d, "\mathrm{id} \otimes S"] \\
 & \Bbbk \arrow[d, "u"] & \\
H \otimes H \arrow[r, "m"'] & H & H \otimes H \arrow[l, "m"]
\end{tikzcd}$$

Example

$R = R_0 \oplus R_1$: super alg.

R_0

$$GL_{m|n}(R) = \left\{ \begin{array}{cc} \left(\begin{array}{c|c} \boxed{X} & \boxed{P} \\ \hline \boxed{Q} & \boxed{Y} \end{array} \right) & \begin{array}{l} \}m \\ \}n \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{m} & \underbrace{\quad}_{n} \end{array} & \end{array} \in GL_{m+n}(R) \right\}$$

R_1

このとき $k[GL_{m|n}(R)]$ は super alg. & Hopf alg. □

■ Super manifold

k : complete field

super algebra は super commutative $(ab = (-1)^{|a||b|} ba)$ を仮定する.

X : manifold

$\mathcal{F} = \mathcal{F}_X$: X上の正則関数の層

Definition

(i) $X = (|X|, \mathcal{O}_X)$ が super ringed space

$\overset{\text{def}}{\Longleftrightarrow}$ ・ $|X|$: top. sp

・ $\mathcal{O}_X$: sheaf of super alg. (i.e.) $U \overset{\text{open}}{\subset} |X|$, $\mathcal{O}_X(U)$: super alg.

super ringed space の category を SRS とする.

(ii) X : manifold に対して, $(|X|, \mathcal{F}_X \otimes \wedge(W))$ (ただし, W : purely odd, $\dim(W) < \infty$)

を strongly split super manifold という.

その category を SSMFD とする.

(iii) $Z = (|Z|, \mathcal{O})$ が super manifold

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists \{U_i\}_{i \in I}$: open covering of $|Z|$ (s.t.) $(|U|, \mathcal{O}|_U) \in$ SSMFD.

その category を SMFD とする.

MFD $\subset$ SSMFD $\subset$ SMFD $\subset$ SRS.

Definition

G : super Lie group $\Leftrightarrow$ $G \in$ SMFD.
この圏を SLG とかく.

■ Haris - Chandra pair

Definition

(F, V) : Haris-chandra pair
$\overset{def}{\Leftrightarrow}$
- F : Lie group
- V : fin. dim right F-module
- $[-,-] : V \otimes V \longrightarrow \mathrm{Lie}(F)$: F-equiv (s.t.)
  - ① $[v, w] = [w, v]$
  - ② $v \triangleleft [v, v] = 0 \quad (v \in V)$

このとき, $\underbrace{\mathrm{Lie}(F)}_{\ni u} \oplus \underbrace{V}_{\ni v}$ は super Lie alg.
u v に対しては, $[u, v] = -[v, u]$.

HCP : Haris - Chandra pair のなす圏.

Theorem

SLG $\simeq$ HCP / k が
① $k = \mathbb{R}$ のとき成立する. (Kostant '77)
② $k = \mathbb{C}$ のとき成立する (Vishnyakova '10)

Main result [Masuoka - Hoshi - T '20]

k : complete field with $\mathrm{char}(k) \neq 2$.
このとき,

$$\begin{array}{ccc} \mathrm{SLG} & \simeq & \mathrm{HCP} \\ G & \mapsto & (G_{red}, (\mathrm{Lie}\, G)_1). \end{array}$$