

Hasse norm principle for M_{11} extensions

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§ Intro

 K/\mathbb{Q} : Number Field, $n := [K:\mathbb{Q}]$ L : Galois closure $|) \text{ Gal.}$ K $|) n$ \mathbb{Q} $H := \text{Gal}(L/K)$ G/H $G = \text{Gal}(L/\mathbb{Q})$ Question : $G = M_{11}$ とおぼよびた 拡大 K/\mathbb{Q} //拡大 K/\mathbb{Q} の n 個の写像 $\sigma \in N_{K/\mathbb{Q}}$ とかく:

$$N_{K/\mathbb{Q}}(\alpha) := \prod_{\sigma \in G/H} \sigma(\alpha).$$

 $(\alpha_1, \dots, \alpha_m) \in K^m$ に対して

$$N_{K/\mathbb{Q}}(X_1, \dots, X_m) := N_{K/\mathbb{Q}}\left(\sum_{i=1}^m \alpha_i X_i\right) = \prod_{\sigma \in G/H} \sigma\left(\sum_{i=1}^m \alpha_i X_i\right)$$

を K/\mathbb{Q} の n 個と呼ぶ。Question : $c \in \mathbb{Z}$ に対して,

$$\exists? (a_1, \dots, a_m) \in \mathbb{Z}^m \text{ (s.t.) } N_{K/\mathbb{Q}}(a_1, \dots, a_m) = c$$

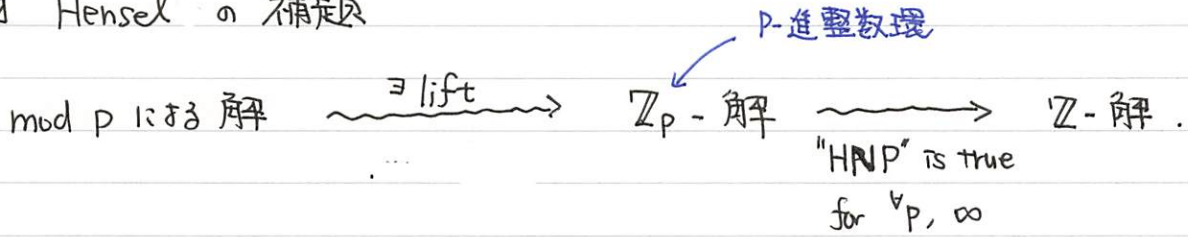
• $m=2$, $K = \mathbb{Q}(\sqrt{D})$, $c=1$.

$$N_{K/\mathbb{Q}}(X, Y) = (X + \sqrt{D}Y)(X - \sqrt{D}Y) = X^2 - DY^2 = 1 \leftarrow \text{Pell's equation}$$

 $\rightsquigarrow \sqrt{D}$ の連分数展開を用いて解を構成できる:• $m=2$, (general)This equation \rightarrow Roth's diophantine application \rightsquigarrow Computer などで解ける.• $m \geq 3$, Only a few specific results.

(e.g.) Schmidt subsp. theorem '79

□ Hensel の補題



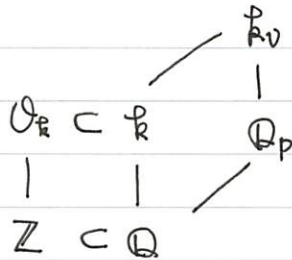
Today: 拡大 K/\mathbb{R} で $\text{Gal}(K/\mathbb{R}) \simeq M_{11}$ のとき, HNP はいつ成立するかを決定する.

§2 HNP

\mathbb{R} : 代数体

\mathbb{R}_v : \mathbb{R} の v に対する完備化 $\rightsquigarrow \mathbb{R} \hookrightarrow \mathbb{R}_v$ という埋め込みがある.

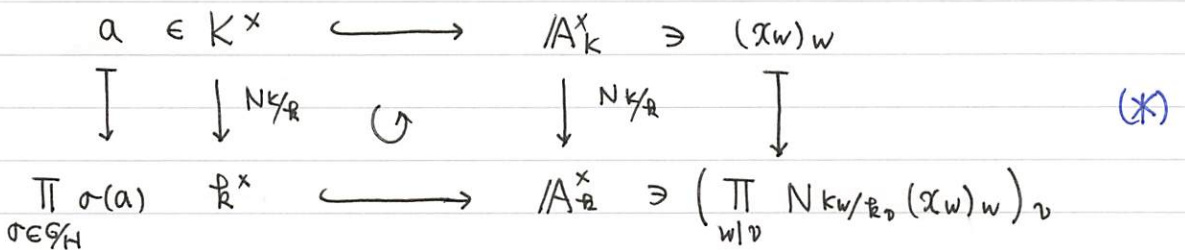
$\rightsquigarrow \mathbb{R} \hookrightarrow \prod_{v \in V_{\mathbb{R}}} \mathbb{R}_v$
 $v \in V_{\mathbb{R}} \hookrightarrow V_{\mathbb{R}}$: place 全体.
 $a \mapsto (a, a, \dots)$



$\mathbb{A}_{\mathbb{R}} := \{ (a_v) \in \prod_v \mathbb{R}_v \mid v(a_v) \geq 0 \text{ for almost all } v \}$
 \uparrow アデール環

\cup
 $\mathbb{A}_{\mathbb{R}}^{\times}$: idele group of \mathbb{R} .

• K/\mathbb{R} を代数体の有限次拡大とする.



Definition

$$\text{Obs}(K/\mathbb{R}) := \frac{\mathbb{R}^\times \cap N_{K/\mathbb{R}}(A_K^\times)}{N_{K/\mathbb{R}}(K^\times)}$$

← local な情報 (numerator)
 ← global な情報 (denominator)

図式(*)から得られる
 local な条件と, その $N_{K/\mathbb{R}}$ の
 差をいっている.

• Hasse norm principle (HNP) holds for $K/\mathbb{R} \stackrel{\text{def}}{\iff} \text{Obs}(K/\mathbb{R}) = 1$

Remark

HNP holds for $K/\mathbb{R} \stackrel{\text{iff}}{\iff}$ "HP" holds for $N_{K/\mathbb{R}}(X_1, \dots, X_n) = C \ (\forall C \in \mathbb{R}^\times)$

Hasse principle. (\exists local solutions for $\forall p \implies \exists$ global sol)

Theorem 1 [Hasse norm theorem '31]

L/\mathbb{R} : cyclic extension (i.e.) $\text{Gal}(L/\mathbb{R}) \simeq \text{cyclic gp } C_n \implies \text{Obs}(L/\mathbb{R}) = 1 \quad //$

- $G = V_4 = C_2 \times C_2$ ← Not cyclic.
 $\text{Obs}(\mathbb{Q}(\sqrt{-3}, \sqrt{3})/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}$.) obs は体には依存している.
 $\text{Obs}(\mathbb{Q}(\sqrt{2}, \sqrt{-1})/\mathbb{Q}) = 1$

• K/\mathbb{R} : Galois ext $\implies \text{Obs}(K/\mathbb{R}) = \ker \{ H^3(G, \mathbb{Z}) \xrightarrow{\text{res}} \bigoplus_{\mathbb{P}} H^3(G_{\mathbb{P}}, \mathbb{Z}) \}$
 $\mathbb{P} = \mathbb{R}, \mathbb{C}$ $G_{\mathbb{P}} := \{ \sigma \in G \mid \sigma(p_{\mathbb{P}}) = p_{\mathbb{P}} \}$
 ([Tate '67])

§3 Norm one tori.

\mathbb{R} : a field

T : algebraic \mathbb{R} -torus (i.e.) T : alg. gp (s.t.) $T \times_{\mathbb{R}} \bar{\mathbb{R}} \simeq (\mathbb{G}_{m, \bar{\mathbb{R}}})^n$

$(\exists L/\mathbb{R}$: finite (s.t.) Galois $T \times_{\mathbb{R}} L \simeq (\mathbb{G}_{m, L})^n$) $\stackrel{\parallel}{\simeq} \frac{\mathbb{R}^\times}{\mathbb{R}}$

Theorem 2 [Ono '69]

T : alg. \mathbb{R} -torus, split / $(L) \xleftrightarrow{1:1} M$: G -lattice
 $T \xrightarrow{\quad} \hat{T} := \text{Hom}(T(L), L^\times)$

• K/\mathbb{R} : fin. ext :

$$\begin{array}{ccccccc} 0 & \rightarrow & I_{G/H} & \rightarrow & \mathbb{Z}[G/H] & \xrightarrow{\varepsilon} & \mathbb{Z} \rightarrow 0 \\ & & \text{ii} & & \cup & & \cup \\ & & \ker(\varepsilon) & & \sum_{\sigma \in G/H} a_{\sigma} \tau & \longmapsto & \sum_{\sigma} a_{\sigma} \end{array}$$

$\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$
 $\xrightarrow{\sim}$

$$0 \leftarrow J_{G/H} \leftarrow \mathbb{Z}[G/H] \leftarrow \mathbb{Z} \leftarrow 0$$

Thm 2
 $\xrightarrow{\sim}$

$$1 \rightarrow R_{K/\mathbb{R}}^{(1)}(G_{m,k}) \rightarrow R_{K/\mathbb{R}}(G_{m,k}) \rightarrow G_{m,\mathbb{R}} \rightarrow 1$$

↑ Weil restriction

• $\Omega(T) := \ker (H^1(\mathbb{R}, T) \rightarrow \bigoplus_{\mathbb{v}} H^1(\mathbb{R}_{\mathbb{v}}, T))$: Tate - Shafarevich group.

• $A(T) := \left(\prod_{\mathbb{v}} T(\mathbb{R}_{\mathbb{v}}) \right) / \overline{T(\mathbb{R})}$: kernel of weak approximation.

Theorem 3 [Ono '63]

$$\text{Obs}(K/\mathbb{R}) \simeq \Omega(R_{K/\mathbb{R}}^{(1)}(G_{m,k}))$$

Theorem 4 [Voskresenskii '69]

T : alg. \mathbb{R} -torus

X : smooth projective model of T (\mathbb{R} が正標数 のときは sm. \mathbb{R} -compactification)

このとき,

$$0 \rightarrow A(T) \rightarrow H^1(\mathbb{R}, \text{Pic } \bar{X})^{\vee} \rightarrow \Omega(T) \rightarrow 0$$

G -lattice の理論で計算できる. [HY'12]

特に,

$$H^1(\mathbb{R}, \text{Pic } \bar{X}) = 0 \Rightarrow A(T) = \Omega(T) = 0.$$

§4 Main result.

$$G := \text{Gal}(L/\mathbb{R}) \cong M_{11} \quad \leftarrow \#M_{11} = 7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$$

$$H := \text{Gal}(L/k) \not\cong G$$

\leftarrow 39-cases up to conj.

$$T := R_{k/\mathbb{R}}^{(1)}(Gm, k)$$

Theorem 5 [Hoshi-K-Yamasaki '22]

26-cases

$$H^1(\mathbb{R}, \text{Pic } \bar{X}) = \begin{cases} 0 & \text{if } \#H \text{ is odd or } \text{Syl}_2(H) \text{ is Not cyc.} \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise. } \leftarrow \end{cases}$$

13 cases

Remark

$\text{Syl}_2(G) = QD_8$ である, $\text{Syl}_2(H)$ が cyclic のときは C_2, C_4, C_8 のいずれか.

Theorem 6 [H-K-Y]

$$(1) \#H: \text{even} \ \& \ \text{Syl}_2(H) \neq C_2, C_4, C_8 \Rightarrow A(T) = \text{Ш}(T) = 0$$

$$(2) \text{—————} \ \text{Syl}_2(H) = C_2, C_4, C_8 \Rightarrow \text{Ш}(T) = 0$$

$$\Leftrightarrow \left\{ \begin{array}{l} \exists v: \text{place of } \mathbb{R} \text{ (s.t.)} \\ V_4 \leq G_v \text{ or } Q_8 \leq G_v \text{ if } \text{Syl}_2(H) = C_2 \\ D_4 \leq G_v \text{ or } Q_8 \leq G_v \text{ if } \text{Syl}_2(H) = C_4 \\ QD_8 \leq G_v \text{ if } \text{Syl}_2(H) = C_8. \end{array} \right.$$