Hasse norm principle for M_{11} extensions 金井（新潟）

§ Intro

$K/\mathbb{Q}$: Number Field , $n := [K:\mathbb{Q}]$

$$\begin{array}{ll} L & : \text{Galois closure} \\ | &) \text{Gal.} \quad H := \mathrm{Gal}(L/K) \\ K & \qquad\qquad\qquad\qquad G = \mathrm{Gal}(L/\mathbb{Q}) \\ | &) n \quad G/H. \\ \mathbb{Q} & \end{array}$$

Question : $G = M_{11}$ となるような 拡大 $K/\mathbb{Q}$ //

拡大 $K/\mathbb{Q}$ のノルム写像を $N_{K/\mathbb{Q}}$ とかく：

$$N_{K/\mathbb{Q}}(\alpha) := \prod_{\sigma \in G/H} \sigma(\alpha).$$

$(\alpha_1, \cdots, \alpha_m) \in K^m$ に対して

$$N_{K/\mathbb{Q}}(X_1, \cdots, X_m) := N_{K/\mathbb{Q}}\Big(\sum_{i=1}^{m} \alpha_i X_i\Big) = \prod_{\sigma \in G/H} \sigma\Big(\sum_{i=1}^{m} \alpha_i X_i\Big)$$

を $K/\mathbb{Q}$ の ノルム と呼ぶ.

Question : $c \in \mathbb{Z}$ に対して,

$$\exists ? \ (a_1, \cdots, a_m) \in \mathbb{Z}^m \ \ (\text{s.t.}) \ \ N_{K/\mathbb{Q}}(a_1, \cdots, a_m) = c$$

- $m = 2$, $K = \mathbb{Q}(\sqrt{D})$, $c = 1$.
  $N_{K/\mathbb{Q}}(X, Y) = (X + \sqrt{D}Y)(X - \sqrt{D}Y) = X^2 - DY^2 = 1$ ← Pell's equation
  $\rightsquigarrow$ $\sqrt{D}$ の連分数展開を用いて解を構成できる：

- $m = 2$, (general)
  Thue equation $\longrightarrow$ Roth's diophantine application
  $\rightsquigarrow$ Computer などで解ける .

- $m \geq 3$, Only a few specific results.
  (e.g.) Schmidt subsp. theorem '71

▨ Hensel の補題

$$\text{mod } p \text{ による解} \xrightarrow[\cdots]{\exists\, \text{lift}} \underset{\text{p-進整数環}}{\mathbb{Z}_p}\text{-解} \xrightarrow[\substack{\text{“HNP” is true} \\ \text{for } {}^\forall p,\ \infty}]{} \mathbb{Z}\text{-解}.$$

Today: 拡大 K/k で $\mathrm{Gal}(K/k) \simeq M_{11}$ のとき, HNPはいつ成立するかを決定する.

§2 HNP

k: 代数体

k_v: k の v における完備化 $\rightsquigarrow$ $k \hookrightarrow k_v$ という埋め込みがある.

$\rightsquigarrow$ $k \hookrightarrow \prod_{v \in V_k} k_v$ ← V_k: place全体.

$a \longmapsto (a, a, \cdots)$

$$\begin{array}{ccccc} & & & & k_v \\ & & & \diagup & | \\ \mathcal{O}_k & \subset & k & & \mathbb{Q}_p \\ | & & | & \diagup & \\ \mathbb{Z} & \subset & \mathbb{Q} & & \end{array}$$

$\mathbb{A}_k := \{ (a_v) \in \prod_v k_v \mid v(a_v) \geq 0 \text{ for almost all } v \}$

↑ アデール環

$\cup$

$\mathbb{A}_k^\times$: Idele group of k.

・K/k を代数体の有限次拡大とする.

$$\begin{array}{ccccccc} a & \in & K^\times & \longrightarrow & \mathbb{A}_K^\times & \ni & (x_w)_w \\ \downarrow & & \downarrow N_{K/k} & \circlearrowright & \downarrow N_{K/k} & & \downarrow \\ \prod_{\sigma \in G/H} \sigma(a) & & k^\times & \longrightarrow & \mathbb{A}_k^\times & \ni & \left(\prod_{w \mid v} N_{K_w/k_v}(x_w)_w \right)_v \end{array} \qquad (*)$$

Definition

・ $\mathrm{Obs}(K/k) := \dfrac{k^\times \cap N_{K/k}(\mathbb{A}_K^\times)}{N_{K/k}(K^\times)}$

local な情報
global な情報.

図式(*)から得られる local なデータと、もとの $N_{K/k}$ の差をはかっている.

・ Hasse norm principle (HNP) holds for $K/k \overset{\text{def}}{\iff} \mathrm{Obs}(K/k) = 1$

Remark

Hasse principle. ($\exists$ local solution for $\forall p$ $\Rightarrow$ $\exists$ global sol)

HNP holds for $K/k \overset{\text{iff}}{\iff}$ "HP" holds for $N_{K/k}(X_1, \cdots, X_n) = c$ ($\forall c \in k^\times$)

Theorem 1 [Hasse norm theorem '31]

L/k : cyclic extension (i.e.) $\mathrm{Gal}(L/k) \simeq$ cyclic gp C_n $\Rightarrow \mathrm{Obs}(L/k) = 1$ //

・ $G = V_4 = C_2 \times C_2$ ← Not cyclic.

$\mathrm{Obs}(\mathbb{Q}(\sqrt{-39}, \sqrt{-3})/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}$.

$\mathrm{Obs}(\mathbb{Q}(\sqrt{2}, \sqrt{-1})/\mathbb{Q}) = 1$

obs は体に依存している.

・ K/k : Galois ext $\Rightarrow \mathrm{Obs}(K/k) = \ker\{H^3(G, \mathbb{Z}) \xrightarrow{\text{res}} \bigoplus_v H^3(G_v, \mathbb{Z})\}$

ただし $G_v := \{\sigma \in G \mid \sigma(p_v) = p_v\}$

([Tate '67])

§3 Norm one tori.

k : a field

T : algebraic k-torus (i.e.) T : alg. gp (s.t.) $T \times_k \bar{k} \simeq (\underbrace{\mathbb{G}_{m,\bar{k}}}_{\| \atop \bar{k}^\times})^n$.

($\exists L/k$: finite Galois (s.t.) $T \times_k L \simeq (\mathbb{G}_{m,L})^n$)

Theorem 2 [Ono '69]

T : alg. k-torus, split $/L$ $\overset{1:1}{\longleftrightarrow}$ M : G-lattice ($G = \mathrm{Gal}(L/k)$)

$T \longmapsto \hat{T} := \mathrm{Hom}(T(L), L^\times)$

・ K/k : fin. ext :

$$0 \to I_{G/H} \to \mathbb{Z}[G/H] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$$I_{G/H} = \ker(\varepsilon), \quad \mathbb{Z}[G/H] \ni \sum_{\sigma \in G/H} a_\sigma \sigma \longmapsto \sum_\sigma a_\sigma \in \mathbb{Z}$$

$\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ ⇝ $$0 \leftarrow J_{G/H} \leftarrow \mathbb{Z}[G/H] \leftarrow \mathbb{Z} \leftarrow 0$$

Thm 2 ⇝ $$1 \to R^{(1)}_{K/k}(\mathbb{G}_{m,K}) \to R_{K/k}(\mathbb{G}_{m,K}) \to \mathbb{G}_{m,k} \to 1$$

Weil restriction

・ $Ш(T) := \ker\left(H^1(k, T) \to \bigoplus_v H^1(k_v, T) \right)$: Tate - Shafarevich group.

・ $A(T) := \left(\prod_v T(k_v) \right) / \overline{T(k)}$: kernel of weak approximation.

Theorem 3 [Ono '63]

$\mathrm{Obs}(K/k) \simeq Ш(R^{(1)}_{K/k}(\mathbb{G}_{m,K}))$

Theorem 4 [Voskresenskii '69]

T : alg. k-torus

X : smooth projective model of T （kが正標数のときは sm. k-compactification）

このとき,

$$0 \to A(T) \to H^1(k, \mathrm{Pic}\, \bar{X})^\vee \to Ш(T) \to 0.$$

G-lattice の理論で計算できる. [HY'12]

特に,

$$H^1(k, \mathrm{Pic}\, \bar{X}) = 0 \Rightarrow A(T) = Ш(T) = 0.$$

§4 Main result.

· $G := \mathrm{Gal}(L/k) \simeq M_{11}$. $\#M_{11} = 7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$

$H := \underbrace{\mathrm{Gal}(L/K)}_{\text{39-cases up to conj.}} \lneq G$

$T := R^{(1)}_{K/k}(\mathbb{G}_{m,K})$

Theorem 5 [Hoshi - K - Yamasaki '22]

$$H^1(k, \mathrm{Pic}\,\bar{X}) = \begin{cases} 0 & \text{if } \#H \text{ is odd or } \mathrm{Syl}_2(H) \text{ is Not cyc.} \quad \text{26-cases} \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise.} \quad \text{13 cases} \end{cases}$$

Remark

$\mathrm{Syl}_2(G) = QD_8$ なので、 $\mathrm{Syl}_2(H)$ が cyclic のときは C_2, C_4, C_8 のいずれか.

Theorem 6 [H-K-Y]

(1) $\#H$: even & $\mathrm{Syl}_2(H) \neq C_2, C_4, C_8 \Rightarrow A(T) = Ш(T) = 0$

(2) ———— $\mathrm{Syl}_2(H) = C_2, C_4, C_8 \Rightarrow Ш(T) = 0$

$$\overset{\text{iff}}{\Longleftrightarrow} \begin{cases} \exists v : \text{place of } k \text{ (s.t.)} & \\ V_4 \leq G_v \text{ or } Q_8 \leq G_v & \text{if } \mathrm{Syl}_2(H) = C_2 \\ D_4 \leq G_v \text{ or } Q_8 \leq G_v & \text{if } \mathrm{Syl}_2(H) = C_4 \\ QD_8 \leq G_v & \text{if } \mathrm{Syl}_2(H) = C_8. \end{cases}$$