§1　Intro（quotient sing）

§2　2-dim'l（Gorenstain）quotient sing's

§3　Higher dimensions

§1 :

▨ Group actions

$G \subseteq SL_n(\mathbb{C})$: fin. Aberian subgroup with $|G| = r$.

今, Gの元たちは可換なので 同時対角化できる:

$$g = \mathrm{diag}\left(e^{2\pi\sqrt{-1}\frac{a_1}{r}}, \dots, e^{2\pi\sqrt{-1}\frac{a_n}{r}}\right) =: \frac{1}{r}(a_1, a_2, \dots, a_n)$$

もし, $G = \langle \frac{1}{r}(a_1, \dots, a_n) \rangle$ ならば, G を $\frac{1}{r}(a_1, \dots, a_n)$ で書き表す.

群の作用 $G \overset{\sigma}{\curvearrowright} \mathbb{C}^n$ は

$$\begin{array}{rccc} \sigma: & G \times \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (g, x) & \longmapsto & gx = \left(e^{2\pi\sqrt{-1}\frac{a_1}{r}}x_1, \dots, e^{2\pi\sqrt{-1}\frac{a_n}{r}}x_n\right) \end{array}$$

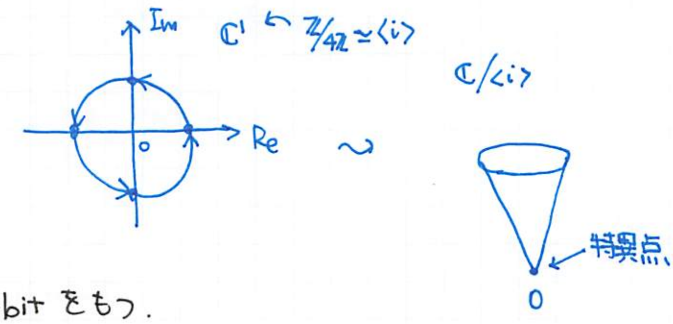
で定義される.

▨ quotients.

Definition 1

代数多様体の射 $\phi: \mathbb{C}^n \longrightarrow \mathbb{C}^n/G$ が geometric quotient であるとは,

①
$$\begin{tikzcd} G \times \mathbb{C}^n \arrow[r, "\sigma"] \arrow[d, "射影\ \pi_2"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{C}^n \arrow[d, "\phi"] \\ \mathbb{C}^n \arrow[r, "\phi"'] & \mathbb{C}^n/G \end{tikzcd}$$



② $\forall y \in \mathbb{C}^n/G$ に対して, $\phi^{-1}(y)$ は 只1つの orbit をもつ.

③ $U \overset{\text{open}}{\subseteq} \mathbb{C}^n/G \implies \phi^{-1}(U)$ は open

④
$$\begin{tikzcd} G \times \phi^{-1}(U) \arrow[r, "\sigma"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\circlearrowright"] & \phi^{-1}(U) \arrow[d, "f"] \\ \phi^{-1}(U) \arrow[r, "f"'] & \mathbb{C} \end{tikzcd} \quad \overset{\text{iff}}{\Longleftrightarrow} \quad f \in \Gamma(U, \phi_* \mathcal{O}_{\mathbb{C}^n}) \subset \Gamma(U, \mathcal{O}_{\mathbb{C}^n/G})$$

Notation 2

$G = \frac{1}{r}(a_1, \cdots, a_n)$ のとき，$\mathbb{C}^n/G$ は type $\frac{1}{r}(a_1, \cdots, a_n)$ であるという．

Proposition 3

$G = \frac{1}{r}(a_1, \cdots, a_n)$ として，$(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ を Analitic space とする．

このとき

$$\exists\, (\phi,\ \mathbb{C}^n/G) \quad \text{and} \quad \mathcal{O}_{\mathbb{C}^n/G} = \phi_* \mathcal{O}_{\mathbb{C}^n}^G$$

Example 4

$G = \frac{1}{2}(1,1)$ ，$R = \mathbb{C}[x,y]$ とすれば，$R^G = \mathbb{C}[\overset{a}{x^2}, \overset{b}{xy}, \overset{c}{y^2}] \simeq \mathbb{C}[a,b,c]/(ac-b^2)$

$$\left(\begin{array}{c} G \curvearrowright \mathbb{C}^2 \\ \shortparallel \\ \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \end{array} \right)$$

▨ Class of quot. sing.

Definition 5

特異点 (X, x) が Gorenstain とは

- $\underset{\sim\sim}{\omega_X}$: invertible & CM $\overset{\text{iff}}{\Longleftrightarrow}$ $\mathcal{O}_{X,x}$: Gorenstein ring

特異的の germ

Definition 6

K_X : a canonical divisor $\overset{\text{def}}{\Longleftrightarrow}$ $\omega_X \simeq \mathcal{O}_X(K_X)$

Definition 7

$f : Y \to X$: a resolution

$E_1, \ldots, E_r$: the exceptional div.

このとき

$$K_Y = f^* K_X + \sum_{i=1}^{r} \underset{\underset{\mathbb{Q}}{\cap}}{a_i} E_i .$$

(X, x) : a canonical sing $\overset{\text{def}}{\iff}$ $a_i \geq 0$
(terminal) ($a_i > 0$)

Definition 8

$f : Y \to X$: a minimal resolution $\overset{\text{def}}{\iff}$ $\forall g : Y' \to X$: a resolution

$$\begin{tikzcd} Y' \arrow[rr, "g"] \arrow[dr, dashed, "\varphi"'] & \arrow[d, phantom, "\circlearrowleft"] & X \\ & Y \arrow[ur, "f"'] & \end{tikzcd}$$

§2 :

▨ 2-dim'l (Gor) quot. sing.

Assumption 9

$G \subseteq SL(n, \mathbb{C})$: small (e.g.)

Example 10

$$\left[\begin{array}{l} G = \frac{1}{2}(0,0,1) \text{ なら } R = \mathbb{C}[x,y,z] \text{ に対して} \\ R^G = \mathbb{C}[x,y,z^2] \simeq \mathbb{C}[x,y,z]. \\ \rightsquigarrow \mathbb{C}^3/G \simeq \mathbb{C}^3 \end{array}\right]$$

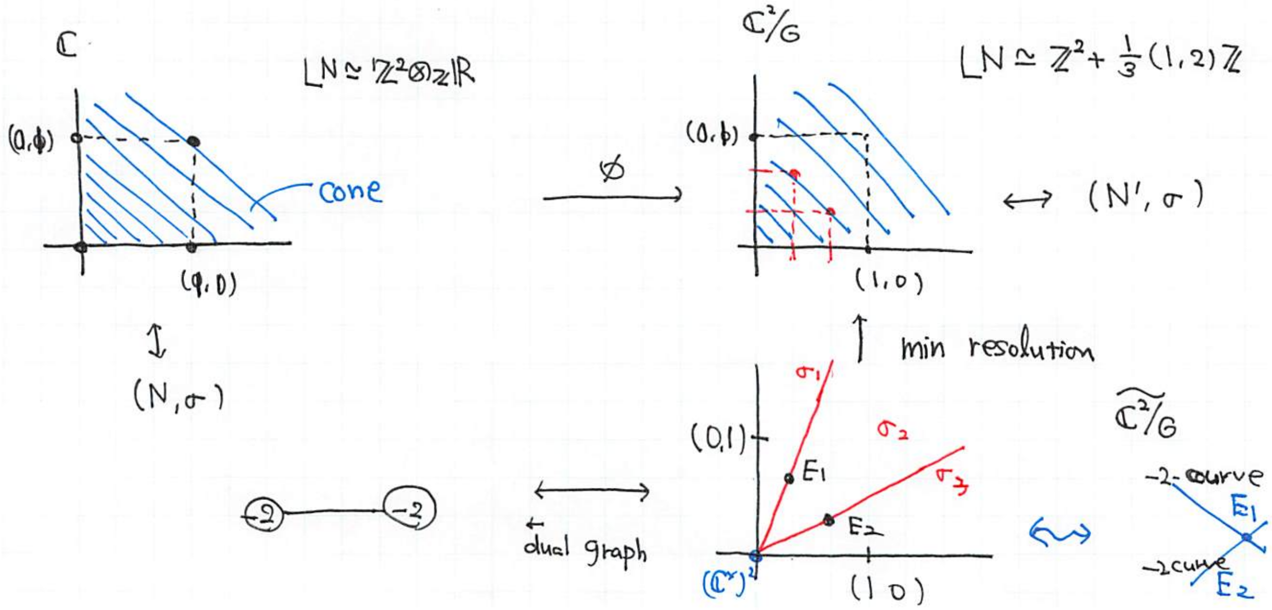
という状況を除く.

Proposition 11

(X, x) : 2-dim'l Gor-sing $\Rightarrow$ (X, x) has a min resol. $f : Y \to X$ and $K_Y = f^* K_X$.

Example 12

$\mathbb{C}^2/G : \frac{1}{3}(1,2)$



一方、 $w = e^{2\pi\sqrt{-1}\frac{1}{3}}$ とすれば

$$G = \langle \underbrace{\begin{pmatrix} w^1 & 0 \\ 0 & w^2 \end{pmatrix}}_{= g} \rangle$$ ~~[illegible]~~ 2

とすると

$\rho_N : G \hookrightarrow SL(2, \mathbb{C}) \subset GL(2, \mathbb{C})$ the Natural rep.

χ_N : the character of ρ_N.

$\underset{\text{自明表現}}{\underset{\|}{\rho_0}}, \rho_1, \rho_2$: irreducible rep. of G

$\rightsquigarrow \quad \rho_N(g^i) = \begin{pmatrix} w^i & 0 \\ 0 & w^{-i} \end{pmatrix} \qquad \chi_N(g^i) = w^i + w^{-i}$

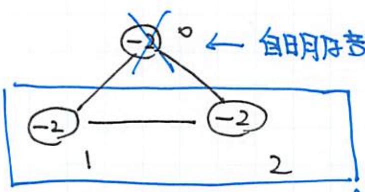
$\rho_k(g^i) = w^{ki} \qquad \chi_k(g^i) = w^{ki}$

$$\rho_N \otimes \rho_j = \bigoplus_{j=0}^{2} n_{ij}\, \rho_j$$

$\rightsquigarrow \quad C = (n_{ij} - 2\delta_{ij})$: the extended dinkin

$$= \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$\rightsquigarrow$



← 自明な部分を削る.

先程の Dynkin diagram (Mackay correspondence)

On the other hand, Hirzebruch - Jung conn. frac.

$\frac{1}{3}(1, 2) \rightsquigarrow \frac{3}{2} = 2 - \frac{1}{2} \rightsquigarrow$ (-2)——(-2)

Ⅷ Macay correspondence

Fact

一般に $\nexists$ min. res. for $\mathbb{C}^n/G$.

$\rightsquigarrow$

$K_Y = f^*K_X$ をもつ resolution に着目.

Definition 13

$f : Y \longrightarrow X$ (Gor sing)

(f, Y) : Crepant $\overset{\text{def}}{\Longleftrightarrow}$ $K_Y = f^*K_X$.

Theorem 14 [Bridgeland, King, Reid, .. 2000]

the derived macay corresp.

$\mathbb{C}^3/G$: Gor. quotient sing.

$Y =$ G-Hib$(\mathbb{C}^3)$ is a projective crepant resol:, and $D^b(\mathrm{coh}(Y)) \overset{\text{tri}}{\xrightarrow{\sim}} D^b(\mathrm{coh}^G(\mathbb{C}^3))$

Theorem 15 [Kawamata '2014]

$\mathbb{C}^n/G$: a Gor. ab quotient. sing

もし, $\exists$ a crepant resol $\Longrightarrow$ DMC holds

Ⅸ Ashikaga conn. frac.

Definition 16

$\frac{\mathrm{a}}{r}$ $(\mathrm{a} \in \mathbb{Z}^n_{\geq 0})$ で $r \in \mathbb{N}$, $0 \leq a_i \leq r-1$ とする. このとき

$$\frac{\mathrm{a}}{r} = \frac{(a_1, a_2, \cdots, a_n)}{r}$$

を proper fraction という.

Definition 17

semi-unimodular

$0 \leq \overline{a_j}^{a_i} \leq a_i - 1$ mod a_i $\overline{a_j}^{a_i} \equiv a_j \pmod{a_i}$

$$R_i\left(\frac{\mathrm{a}}{r}\right) = \begin{cases} \dfrac{(\overline{1}^{a_i}, \overline{a_2}^{a_i}, \ldots, \overline{a_{i-1}}^{a_i}, \overline{-r}^{a_i}, \overline{a_{i+1}}^{a_i}, \ldots, \overline{a_n}^{a_i})}{a_i} & (a_i \neq 0) \\ \infty & (a_i = 0) \end{cases}$$

$R_i(\infty) = \infty$

Definition 18

R_{i_s}を施していって ∞になる1コ前

$$R_*\left(\frac{a_1}{r}\right) = \frac{a_1}{r} + \sum (R_{i_\ell} \cdots R_{i_1})\left(\frac{a_1}{r}\right) x_{i_1} \cdots x_{i_\ell}$$ を reminder polynomial という.

\+

round down polynomial

足利連分数展開.

Fact

足利連分数展開 は H-J conn frac の 次元 に関する拡張である.

また これは Fujiki - Oka resolution と呼ばれる toric resolution を与えている.

Theorem [S, Y. Sato]

① $\exists$ crepant iterated Fujiki - Oka resolution for any Gor ab. sing.

② 3次元の cannonical cyclic quotient sing に対して $\exists$ Hilbert basis resol

(s.t.) これが iteretaed F-O resol を与える.